ON THE EXISTENCE OF THE STIELTJES INTEGRAL*

BY

H. L. SMITH

Lebesgue† has shown how to treat the Riemann integral by studying the content of an associated point set. The present paper is an attempt in the same direction for the Stieltjes integral. A pair of conditions are found which are necessary and sufficient for the existence of the integral, one of which concerns itself with the associated point set. The other is automatically satisfied for a large class of integrals comprising (1) those for which the associated point set is a (continuous) curve with at most a finite number of multiple points; and (2) those for which the measure function is of limited variation. A consequence is that a simple closed curve must be squarable if its line integral $\int y\,dx$ exists. Among the examples given is one which shows that a simple closed curve may be squarable and still fail to have an existent line integral $\int y\,dx$.

1. **Definitions and notations.** If I', I'' are two sub-intervals of T, $I' \cdot I''$ will denote the interval common to I', I''.

A general partition $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = 1$ of the interval T: $0 \leqq t \leqq 1$ will be denoted by the notation π; a partition of a sub-interval I of T will be denoted by $\pi(I)$. A general cell $t_i \leqq t \leqq t_{i+1}$ of a partition π will be denoted by $\Delta\pi$. The symbol $\sum_{\Delta\pi}^{\pi}$ will denote a summation over all the cells $\Delta\pi$ of a partition π. The upper (greater) and lower (lesser) end points of an interval I will be denoted by $\overline{I}$, $\underline{I}$ respectively, and t^I will denote any point of an interval $\underline{I}$. By $\pi' \times \pi''$ will be denoted the partition consisting of all non-singular $\Delta\pi' \cdot \Delta\pi''$.

Every numerically-valued function $\theta(t)$ defined on T gives rise to an associated function $\theta(I)$ on the class of all sub-intervals I of T defined by the equation

$$\theta(I) \equiv \theta(\overline{I}) - \theta(\underline{I}).$$

The symbols $S_\pi \psi \Delta \varphi$ and $S_\pi^0 \; \psi \Delta \varphi$ are defined by the equations

$$S_\pi \psi \Delta \varphi \equiv \sum_{\Delta\pi}^{\pi} \psi(t^{\Delta\pi}) \varphi(\Delta\pi),$$

$$S_\pi^0 \psi \Delta \varphi \equiv \sum_{\Delta\pi}^{\pi} \frac{1}{2} \{\psi(\underline{\Delta\pi}) + \psi(\overline{\Delta\pi})\} \varphi(\Delta\pi);$$

* Presented to the Society, December 29, 1923.

† *Leçons sur l'Intégration*, 1904, p. 45.

and are respectively multiply-valued and singly-valued functions of π for given φ, ψ.

A partition π_1 is *finer* than a partition π_2, in notation $\pi_1 \mathrm{F} \pi_2$, if every cell $\Delta\pi_1$ of π_1 lies entirely in some cell $\Delta\pi_2$ of π_2. The binary relation F is transitive, reflexive and has the composition property as defined by E. H. Moore and the author.* It, therefore, serves to define a limit process $\mathrm{L_F}$ applicable to numerically-valued functions $\theta(\pi)$ defined for the class of all partitions of T. Thus $\mathrm{L_F}\ \theta = a$ provided there exists a system $(\pi_e \mid e)$ $(e > 0)$ such that

$$|\theta(\pi) - a| \leqq e \qquad (\pi \mathrm{F} \pi_e) \qquad (e > 0).$$

Every partition π has a norm $\mathrm{N}\pi$ defined as the length $\overline{\Delta\pi} - \underline{\Delta\pi}$ of its longest cell $\Delta\pi$. This gives rise to a second limit process $\mathrm{L_N}$ on functions $\theta(\pi)$ defined as follows: $\mathrm{L_N}\theta = a$ provided there exists a system $(d_e \mid e)$ such that

$$|\theta(\pi) - a| \leqq e \qquad (\pi,\ \mathrm{N}\pi \leqq d_e) \qquad (e).$$

The (Riemann) Stieltjes integral $\int_0^1 \psi(t)\, d\varphi(t)$ or more briefly $\int_0^1 \psi\, d\varphi$ exists in the sense

$$\begin{matrix} (FS) \\ (FW) \\ (NS) \\ (NW) \end{matrix}$$

provided

$$\begin{matrix} \mathrm{L_F}\, S_\pi\, \psi \Delta \varphi \\ \mathrm{L_F}\, S_\pi^0\, \psi \Delta \varphi \\ \mathrm{L_N}\, S\ \psi \Delta \varphi \\ \mathrm{L_N}\, S^0\, \psi \Delta \varphi \end{matrix}$$

exists. The same symbol may be used in each of these cases, since, if the integral exists simultaneously in two or more senses, the values are the same. This is easily seen from the following.

Obviously if the integral $\int_0^1 \psi\, d\varphi$ exists in the sense $\begin{smallmatrix} FS \\ NS \end{smallmatrix}$ then it exists (with the same value) in the sense $\begin{smallmatrix} FW \\ NW \end{smallmatrix}$, which justifies the notations S (= strong) and W (= weak). Moreover if the integral exists in the sense $\begin{smallmatrix} NW \\ NS \end{smallmatrix}$ then it exists also in the sense $\begin{smallmatrix} FW \\ FS \end{smallmatrix}$.

* E. H. Moore and H. L. Smith, American Journal of Mathematics, vol. 44 (1922), p. 104.

2. **A necessary and sufficient condition.** The *oscillation* of $S\psi\Delta\varphi$ on I, in notation $O_I S\psi\Delta\varphi$, is defined as the least upper bound of $|S_{\pi'(I)}\psi\Delta\varphi - S_{\pi''(I)}\psi\Delta\varphi|$ for all partitions $\pi'(I)$, $\pi''(I)$ of I (I in T); and $O_I S^0\psi\Delta\varphi$ is similarly defined as the least upper bound of $|S^0_{\pi'(I)}\psi\Delta\varphi - S^0_{\pi''(I)}\psi\Delta\varphi|$. The symbols $O_\pi S\psi\Delta\varphi$, $O_\pi S^0\psi\Delta\varphi$ are defined by the equations

$$O_\pi S\psi\Delta\varphi \equiv \sum_{\Delta\pi}^{\pi} O_{\Delta\pi} S\psi\Delta\varphi,$$

$$O_\pi S^0\psi\Delta\varphi \equiv \sum_{\Delta\pi}^{\pi} O_{\Delta\pi} S^0\psi\Delta\varphi.$$

THEOREM I. *In order that* $\int_0^1 \psi\, d\varphi$ *exist in the sense* (FS), (FW), (NS) *or* (NW) *it is necessary and sufficient that* $\underset{\pi}{\mathrm{L_F}}\, O_\pi S\psi\Delta\varphi = 0$, $\underset{\pi}{\mathrm{L_F}}\, O_\pi S^0\psi\Delta\varphi = 0$, $\underset{\pi}{\mathrm{L_N}}\, O_\pi S\psi\Delta\varphi = 0$ *or* $\underset{\pi}{\mathrm{L_N}}\, O_\pi S^0\psi\Delta\varphi = 0$ *respectively.*

We prove the theorem for the sense (FS).

The condition is *necessary.* For let π be any partition of T and e any positive number. Then there are two partitions $\pi'\mathrm{F}\pi$, $\pi''\mathrm{F}\pi$ such that

$$0 \leqq O_{\Delta\pi} S\psi\Delta\varphi - \{S_{\pi'(\Delta\pi)}\psi\Delta\varphi - S_{\pi''(\Delta\pi)}\psi\Delta\varphi\} \leqq \frac{e}{n},$$

where $\pi'(\Delta\pi)$, $\pi''(\Delta\pi)$ denote respectively the partitions π', π'' as on $\Delta\pi$ and n is the number of cells in π. Hence

$$0 \leqq S_{\pi'}\psi\Delta\varphi - S_{\pi''}\psi\Delta\varphi \leqq O_\pi S\psi\Delta\varphi \leqq e + S_{\pi'}\psi\Delta\varphi - S_{\pi''}\psi\Delta\varphi.$$

On applying the quasi-limit* $\underset{\pi}{L_{\mathrm{F}}}$ to this inequality, there results

$$0 \leqq \underset{\pi}{L_{\mathrm{F}}}\, O_\pi S\psi\Delta\varphi \leqq e \qquad (e)$$

or

$$\underset{\pi}{L_{\mathrm{F}}}\, O_\pi S\psi\Delta\varphi = 0,$$

uniquely; which proves the necessity.†

The condition is *sufficient.* For if $\pi'''\mathrm{F}\pi'$, $\pi'''\mathrm{F}\pi''$, then

$$\begin{aligned} &|S_{\pi'}\psi\Delta\varphi - S_{\pi''}\psi\Delta\varphi| \\ &\qquad \leqq |S_{\pi'}\psi\Delta\varphi - S_{\pi'''}\psi\Delta\varphi| + |S_{\pi'''}\psi\Delta\varphi - S_{\pi''}\psi\Delta\varphi| \\ &\qquad \leqq O_{\pi'}\psi\Delta\varphi + O_{\pi''}\psi\Delta\varphi; \end{aligned}$$

* Moore and Smith, loc, cit., p. 110.

† Moore and Smith, loc. cit., p. 112, Theorem 8.

from which the sufficiency follows on taking the simultaneous F-limit as to π' and π''.

3. **Some necessary conditions.** $O_I\,\psi$, the *oscillation* of ψ on I, is defined as the least upper bound of

$$|\psi(t_1^I) - \psi(t_2^I)|$$

for all t_1^I, t_2^I.

INEQUALITY A. $\quad O_I\,S\,\psi\Delta\varphi \geqq (O_I\,\varphi)\,|\varphi(I)|$.

INEQUALITY A^0. $\quad O_I\,S^0\psi\Delta\varphi \geqq \frac{1}{2}(O_I\,\psi)\,|\varphi(I)|$.

We give the (slightly) more difficult proof, that of A^0. Let $\varphi(I) \neq 0$; the inequality is obvious otherwise. For every $e(>0)$ take t' and t'' in I such that

$$[\psi(t') - \psi(t'')]\ \mathrm{sgn}\,\varphi(I) \geqq O_I\,\psi - e,$$

so that

$$[\psi(t') - \psi(t'')]\,\varphi(I) \qquad \geqq [O_I\,\psi - e]\,|\varphi(I)|.$$

Then

$$\frac{1}{2}[\psi(t') - \psi(t'')]\,\varphi(I) = S_{\pi'(I)}\psi\Delta\varphi - S_{\pi''(I)}\psi\Delta\varphi,$$

where

$$S_{\pi'(I)}\,\psi\Delta\varphi = \frac{\psi(\underline{I}) + \psi(t')}{2}[\varphi(t') - \varphi(\underline{I})] + \frac{\psi(t') + \psi(\bar{I})}{2}[\varphi(\bar{I}) - \varphi(t')],$$

$$S_{\pi''(I)}\,\psi\Delta\varphi = \frac{\psi(\underline{I}) + \psi(t'')}{2}[\varphi(t'') - \varphi(\underline{I})] + \frac{\psi(t'') + \psi(\bar{I})}{2}[\varphi(\bar{I}) - \varphi(t'')];$$

so that

$$S_{\pi'(I)}\psi\Delta\varphi - S_{\pi''(I)}\psi\Delta\varphi \geqq \frac{1}{2}(O_I\,\psi)\,|\varphi(I)| - \frac{1}{2}e|\varphi(I)| \qquad (e).$$

From this inequality A^0 follows.

THEOREM N1. *In order that* $\int_0^1 \psi\,d\varphi$ *exist in the sense* (FS) *or* (FW) *it is necessary that*

$$\underset{\pi}{\mathrm{L_F}}\sum_{\Delta\pi}^{\pi}(O_{\Delta\pi}\psi)\,|\varphi(\Delta\pi)| = 0;$$

in the sense (NS), (NW) *it is necessary that*

$$\underset{\pi}{\mathrm{L_N}}\sum_{\Delta\pi}^{\pi}(O_{\Delta\pi}\psi)\,|\varphi(\Delta\pi)| = 0.$$

This theorem follows at once from inequalities A, A^0.

THEOREM N2. *In order that* $\int_0^1 \psi \, d\varphi$ *exist in the sense* (*NS*) *or* (*NW*) *it is necessary that* φ, ψ *have no simultaneous discontinuities on* T; *in the sense* (*FS*) *that* φ, ψ *have no simultaneous right-hand discontinuities or simultaneous left-hand discontinuities.*

The proof of this theorem is not difficult and is omitted.

4. On the independence of the four senses (*FS*), (*FW*), (*NS*), (*NW*). Of the eleven situations as to the simultaneous existence of $\int_0^1 \psi \, d\varphi$ in different senses indicated in Table 0 the first four are excluded by the fact that (*NS*) implies (*FS*); the next four by the fact that (*FS*) implies (*FW*); the ninth by the fact that (*NS*) implies (*NW*); and the tenth by the fact that (*NW*) implies (*FW*). The last one is excluded by Theorem N2 and Theorem J.

Table 0

	FS	*FW*	*NS*	*NW*
(1)	—	+	+	+
(2)	—	+	+	—
(3)	—	—	+	—
(4)	—	—	+	+
(5)	+	—	+	+
(6)	+	—	+	—
(7)	+	—	—	+
(8)	+	—	—	—
(9)	+	+	+	—
(10)	—	—	—	+
(11)	+	+	—	+

THEOREM J. *If* φ, ψ *have no simultaneous discontinuities, then the existences of* $\int_0^1 \psi \, d\varphi$ *in the senses* (*FS*) *or* (*FW*) *imply existences in the respective senses* (*NS*) *or* (*NW*).

We prove the theorem for the strong senses. We note first that

$$\underset{\pi_0}{\mathrm{L_N}} [S_{\pi_0} \psi \Delta \varphi - S_{\pi \times \pi_0} \psi \Delta \varphi] = 0 \qquad (\pi).$$

Hence

$$(\underset{\pi}{\mathrm{L_F}} \underset{\pi_0}{\mathrm{L_N}}) [S_{\pi_0} \psi \Delta \varphi - S_{\pi \times \pi_0} \psi \Delta \varphi] = 0.$$

Also

$$(\underset{\pi}{\mathrm{L_F}} \underset{\pi_0}{\mathrm{L_N}}) S_{\pi \times \pi_0} \psi \Delta \varphi = \int_0^1 \psi \, d\varphi.$$

Therefore

$$\int_0^1 \psi\, d\varphi = (\underset{\pi}{\mathrm{L}_F}\ \underset{\pi_0}{\mathrm{L}_N})\, S_{\pi_0}\, \psi\, \Delta\varphi = \underset{\pi_0}{\mathrm{L}_N}\, S_{\pi_0}\, \psi\, \Delta\varphi, \qquad \text{Q. E. D.}$$

There remain of the sixteen possible cases only the five exhibited in Table 1. We now give examples to show that these situations actually occur.

Table 1

	FS	FW	NS	NW
I	+	+	+	+
II	+	+	—	—
III	—	+	—	+
IV	—	+	—	—
V	—	—	—	—

Where two examples are given the second (more complicated) one is such that the situation in question holds not only for T but for every I in T.

I. $\varphi(t) = 0, \quad \psi(t) = 0, \quad 0 \leqq t \leqq 1.$

II. $\varphi(t) = 0, \quad 0 \leqq t \leqq \frac{1}{2}; \quad \varphi(t) = 1, \quad \frac{1}{2} < t \leqq 1;$
$\psi(t) = 0, \quad 0 \leqq t < \frac{1}{2}; \quad \psi(t) = 1, \quad \frac{1}{2} \leqq t \leqq 1.$

II′. $\varphi(t) = \sum_{n=1}^{\infty} a_n \epsilon_n'(t), \qquad \psi(t) = \sum_{n=1}^{\infty} a_n \epsilon_n''(t),$

where $a_n = (1/9^{n-1})$ $(n = 1, 2, 3, \cdots)$ and $\epsilon_n'(t) = 1$ when t satisfies one of the inequalities $(3k+1)/3^n \leqq t < (3k+2)/3^n$ $(k = 0, \cdots, 3^{n-1}-1)$, $= 0$ otherwise; and $\epsilon_n''(t) = 1$ when t satisfies one of the inequalities $(3k+1)/3^n < t \leqq (3k+2)/3^n$ $(k = 0, \cdots, 3^{n-1}-1)$, $= 0$ otherwise.

III. $\varphi(t) = \psi(t) = 0, \quad 0 \leqq t < 1; \quad \varphi(1) = \psi(1) = 1.$

III′. $\psi(t) = \varphi(t)$ where $\varphi(t)$ is the parametric representation of the x-coördinate of the Peano-Moore space filling curve as given by E. H. Moore (these Transactions, vol. 1 (1900), p. 80, eq. 27).

IV. $\varphi(t) = \psi(t) = 0, \quad 0 \leqq t \leqq 1, \quad t \neq \frac{1}{2};$
$\varphi(\frac{1}{2}) = \psi(\frac{1}{2}) = 1.$

IV′. $\varphi\left(\frac{2k+1}{2^m}\right) = \psi\left(\frac{2k+1}{2^m}\right) = \frac{1}{4^{m-1}} \qquad (k = 0, \cdots, 2^{m-1}-1);$

$\varphi(t) = \psi(t) = 0, \quad t$ not of form $\frac{2k+1}{2^m}$.

V. $\varphi(t) = \psi(t) = 0$, t rational; $\quad \varphi(t) = \psi(t) = 1$, t irrational.

5. Some lemmas on the operator E. Convex sets. Let α denote a planar set of points. By $\mathrm{E}(\alpha)$ will be denoted the set of all points on closed segments joining pairs of points of α. $\mathrm{E}^2(\alpha) \equiv \mathrm{E}\{\mathrm{E}(\alpha)\}$, etc. If

$E(\alpha) = \alpha$, the set is *convex*. Concerning the operation E the following simple propositions hold:

E 1. If α is connected, $E(\alpha)$ is convex.

E 2. If α is any set, $E(\alpha)$ is connected.

E 3. If α is any set, $E^2(\alpha)$ is convex.

E 4. If α is any set, $E^2(\alpha)$ is the least convex super set of α.

E 5. If α is bounded, $E(\alpha)$ is bounded.

E 6. If α is bounded and closed, $E(\alpha)$ is bounded and closed.

E 7. Every bounded convex set has content.

6. **Some lemmas on the triangles inscribable in a given set.** The *diameter* of a set α is the least upper bound of the distance PQ for all pairs of points P, Q in α.

We shall represent a set and its content (if existent) by the same symbol. If the content is not known to exist, the upper content will be denoted by the symbol for the set with a bar over it.

A triangle PQR is *inscribed* in a set α if the points P, Q, R are in α.

LEMMA T 1. *In any closed bounded convex set α there may be inscribed a $\triangle PQR$ whose area is at least one-fourth the content of α and whose longest side is equal in length to the diameter of α.*

Take two points P, Q of α whose distance apart is equal to the diameter of α. Let p, q be the lines through P, Q respectively and $\perp$ to the line PQ. Clearly all points of α lie between or on the lines p, q. Let P_1P_2, Q_1Q_2 be the segments which are the projections of α on p and q respectively. Then α lies entirely in the rectangle $P_1P_2Q_2Q_1$. Let R_1, R_2 be points of α on P_1Q_1, P_2Q_2 respectively. Let R be that one of the two points $R_1 R_2$ which is at the greater distance from PQ. Then

$$\triangle PQR \geqq \frac{1}{2}(\triangle PQR_1 + \triangle PQR_2) \geqq \frac{1}{4}\text{rectangle } P_1Q_1Q_2P_2 \geqq \frac{\alpha}{4}.$$

LEMMA T 2. *If P, Q are points of $E(\alpha)$ such that PQ equals the diameter of $E(\alpha)$, then P, Q are in α.*

For if one of them, say Q, is not in α it is collinear with two points Q', Q'' in α. But then one of the distances PQ', PQ'' would exceed PQ and PQ would not be a diameter of $E(\alpha)$.

LEMMA T 3. *If P, Q are points of α and R_0 is in $E(\alpha)$, there is a point R of α such that $\triangle PQR \geqq \triangle PQR_0$.*

If R_0 is in α take $R = R_0$. If R_0 is not in α it is collinear with two points R_0', R_0'' of α, one of which is as far from line PQ as R_0 and may be taken as R.

LEMMA T 4. *In any closed bounded set α may be inscribed a $\triangle$ whose area is at least one-fourth the content of* $E^2(\alpha)$.

There is (by Lemma T 1) a $\triangle PQR_0$ inscribed in $E^2(\alpha)$ such that $\triangle PQR \geqq \frac{1}{4}E^2(\alpha)$, and whose longest side, PQ say, is a diameter of $E^2(\alpha)$. By two applications (at most) of Lemma T 2 it is seen that P, Q are in α. By at most two applications of Lemma T 3 a point R in α may be found such that $\triangle PQR \geqq \triangle PQR_0$.

LEMMA T 5. *In any bounded set α there may be inscribed a $\triangle$ whose area is at least one-fifth the upper content of* $E^2(\alpha)$.

This is proved by applying Lemma T 4 to $E^2(\alpha + \alpha')$.

7. **The necessary conditions A.** By $\alpha_{\varphi\psi}(I)$ will be denoted the set of all points $(\varphi(t), \psi(t))$ for t in I.

INEQUALITY B. $O_I S^0 \psi \Delta \varphi \geqq \frac{1}{5} E^2 \alpha_{\varphi\psi}(I)$.

First there are (by Lemma T 5) three points $(\varphi(t_1), \psi(t_1))$, $(\varphi(t_2), \psi(t_2))$, $(\varphi(t_3), \psi(t_3))$ which form a triangle having area exceeding one-fifth the content of $E^2 \alpha_{\varphi\psi}(I)$. That is,

$$\left| \frac{1}{2}\{\psi(t_1)+\psi(t_2)\}[\varphi(t_2)-\varphi(t_1)] + \frac{1}{2}\{\psi(t_2)+\psi(t_3)\}[\varphi(t_3)-\varphi(t_2)] \right.$$
$$\left. + \frac{1}{2}\{\psi(t_3)+\psi(t_1)\}[\varphi(t_1)-\varphi(t_3)] \right| \geqq \frac{1}{5} E^2 \alpha_{\varphi\psi}(I).$$

Now take $\pi' = \underline{I}\, t_1\, t_2\, t_3\, \overline{I}$, $\pi'' = \underline{I}\, t_1\, t_3\, \overline{I}$. Then

$$|S^0_{\pi'} \psi \Delta \varphi - S^0_{\pi''} \psi \Delta \varphi| \geqq \frac{1}{5} E^2 \alpha_{\varphi\psi}(I);$$

from which the inequality follows.

The symbol $S_\pi E^2 \Delta \alpha_{\varphi\psi}$ will denote $\sum^{\pi}_{\Delta\pi} E^2 \alpha_{\varphi\psi}\ (\Delta\pi)$.

THEOREM N 3. *In order that $\int_0^1 \psi\, d\varphi$ exist in the sense (FS) or (FW) it is necessary that*

$$A_F(\varphi\psi): \qquad L_F\, S E^2 \Delta \alpha_{\varphi\psi} = 0;$$

in the sense (NS) or (NW) that

$$A_N(\varphi\psi): \qquad L_N\, S E^2 \Delta \alpha_{\varphi\psi} = 0.$$

COROLLARY. *In order that $\int_0^1 \psi\, d\varphi$ exist in either sense it is necessary that content $\alpha_{\varphi\psi}(T)$ be zero.*

In § 12 will be given an example to show that this is not sufficient even when $\alpha_{\varphi\psi}(T)$ is a simple (continuous) arc.

8. **Lemmas. The relation** U_0. A function $\varphi(t)$ gives rise to an associated function $\varphi_\pi(t)$ relative to π defined as follows:

$$\varphi_\pi(t) = \varphi(\underline{\Delta\pi}) + \varphi(\underline{\Delta\pi}, \overline{\Delta\pi})(t - \underline{\Delta\pi}) \text{ for } t \text{ in } \Delta\pi \text{ and each } \Delta\pi,$$

where

$$\varphi(t_1, t_2) = \frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2}.$$

The following algebraic identity

$$\sum_{i=1}^{n-1}(y_i + y_{i+1})(x_{i+1} - x_i) = (y_1 + y_n)(x_n - x_1) - \sum_{i=2}^{n-1}\begin{vmatrix} x_1 & y_1 & 1 \\ x_i & y_i & 1 \\ x_{i+1} & y_{i+1} & 1 \end{vmatrix}$$

is easily proved by induction. By its aid it is easily shown that

$$S^0_{\pi'}\, \psi_\pi\, d\varphi_\pi = S^0_\pi\, \psi\, \Delta\varphi \qquad (\pi' \mathrm{F}\, \pi),$$

and hence that

$$\int_0^1 \psi_\pi\, d\varphi_\pi = S^0_{\pi'}\, \psi_\pi\, \Delta\varphi_\pi = S^0_\pi\, \psi \Delta\varphi \qquad (\pi' \mathrm{F}\, \pi).$$

$\varphi\, \mathrm{U}_0 \psi$ on I if $\varphi(I')^2 + \psi(I')^2 > 0$ for every I' within I such that $O_{I'}\, \varphi + O_{I'}\, \psi > 0$.

$\varphi\, \mathrm{U}_{00}\, \psi$ on I if there is a $\pi_0(I)$ such that $\varphi\, \mathrm{U}_0\, \psi$ on each $\Delta\pi_0(I)$.

$\varphi\, \mathrm{U}\, \psi$ on I if there is a $\pi_0(I)$ such that for every $\pi(I)\mathrm{F}\,\pi_0(I)$ there is a $\pi'(I)$ such that $\pi'(I)\mathrm{F}\,\pi(I)$ and $\varphi_{\pi'}\, \mathrm{U}_0\, \psi_{\pi'}$ on each $\Delta\pi_0(I)$.

LEMMA $\mathrm{U}_0 1$. *If* $\varphi_\pi\, \mathrm{U}_0\, \psi_\pi$ *on* I *and*

$$\begin{vmatrix} \varphi_\pi(t) & \psi_\pi(t) & 1 \\ \varphi_\pi(\underline{I}) & \psi_\pi(\underline{I}) & 1 \\ \varphi_\pi(\overline{I}) & \psi_\pi(\overline{I}) & 1 \end{vmatrix} \neq 0 \qquad (\underline{I} < t < \overline{I}),$$

then

$$\left| \frac{1}{2}\{\psi_\pi(\underline{I}) + \psi_\pi(\overline{I})\}\, \varphi_\pi(I) - \int_I \psi_\pi\, d\varphi_\pi \right| \leqq \mathrm{E}^2 \alpha_{\varphi_\pi \psi_\pi}(\underline{I}).$$

For if $\pi(I)$ is $t_0 = \underline{I}, t_1, t_2, \cdots, t_{n-1}, t_n = \overline{I}$ and the P_i are the points $(\varphi_\pi(t_i),\ \psi_n(t_i))$ $(i = 0, \cdots, n)$, the polygon $P_0 P_1 \cdots P_n P_0$ is simple and its area the left-hand side of the inequality while the right-hand side is the area of the smallest convex polygon which contains the polygon $P_0 P_1 \cdots P_n P_0$.

9. Reduction of functions to be in the U_0 relation. Some inequalities. Let φ, ψ be continuous on $I(\subseteq T)$. If $\varphi^{-}U_0\psi$ on I, there exists uniquely a sequence $\{G_n\}$ of intervals defined by induction as follows:

(i) G_1 is such that

$$(1)\ G\subseteq I; \qquad (2)\ O_{G_1}\varphi+O_{G_1}\psi>0; \qquad (3)\ \varphi(G_1)^2+\psi(G_1)=0;$$
$$(4)\ \varphi(G)^2+\psi(G)^2>0$$

for every $G\subseteq I$ such that $O_G\varphi+O_G\psi>0$ and such that $\bar{G}-\underline{G}>\bar{G}_1-\underline{G}_1$ or $\bar{G}-\underline{G}=\bar{G}_1-\underline{G}_1$ and $\underline{G}<\underline{G}_1$;

(ii) $G_1,\cdots,G_n$ having been defined, G_{n+1} is such that

$$(0)\ G_{n+1}\cdot(G_1+\cdots+G_n)=0; \quad (1)\ G_{n+1}\subseteq I; \quad (2)\ O_{G_{n+1}}\varphi+O_{G_{n+1}}\psi>0;$$
$$(3)\ \varphi(G_{n+1})^2+\psi(G_{n+1})^2=0; \qquad (4)\ \varphi(G)^2+\psi(G)^2>0$$

for every $G\subseteq I$ such that $G\cdot(G_1+\cdots+G_n)=0$, $O_G\varphi+O_G\psi>0$ and such that $\bar{G}-\underline{G}>\bar{G}_{n+1}-\underline{G}_{n+1}$ or $\bar{G}-\underline{G}=\bar{G}_{n+1}-\underline{G}_{n+1}$ and $\underline{G}<\underline{G}_{n+1}$.

It is clear that $G_i\cdot G_j=0$ for every $i\neq j$, and, therefore, if the sequence is infinite

$$\mathop{\mathrm{L}}_{n}(\bar{G}_n-\underline{G}_n)=0.$$

We may now define two functions $\varphi_{\{\psi I\}}(t)$, $\psi_{\{\varphi I\}}(t)$ over T as follows:

$$\begin{aligned}\varphi_{\{\psi I\}}(t) &= \varphi(t) \text{ if } t \text{ is in } T-\sum_n G_n;\\ &= \varphi(\underline{G}_n)=\varphi(\bar{G}_n) \text{ if } t \text{ is in } G_n \qquad (n);\\ \psi_{\{\varphi I\}}(t) &= \psi(t) \text{ if } t \text{ is in } T-\sum_n G_n;\\ &= \psi(\underline{G}_n)=\psi(\bar{G}_n) \text{ if } t \text{ is in } G_n \qquad (n).\end{aligned}$$

LEMMA U_0 2. *$\varphi_{\{\psi I\}}U_0\psi_{\{\varphi I\}}$ on I.*

Take G so that $O_G\varphi_{\{\psi I\}}+O_G\psi_{\{\varphi I\}}>0$. We are to prove that $\varphi_{\{\psi I\}}(G)^2+\psi_{\{\varphi I\}}(G)^2>0$. We note first that the condition on G implies that G is not entirely in any G_n.

Suppose first that $G\cdot(\sum_n G_n)=0$. Take n so that $\bar{G}-\underline{G}>\bar{G}_n-\underline{G}_n$. This together with $G\subseteq I$, $G\cdot(G_1+\cdots+G_{n-1})=0$ and

$$O_G\varphi+O_G\psi=O_G\varphi_{\{\psi I\}}+O_G\psi_{\{\varphi I\}}>0$$

shows that $\varphi(G)^2+\psi(G)^2>0$. But $\varphi_{\{\psi I\}}(G)^2+\psi_{\{\varphi I\}}(G)^2=\varphi(G)^2+\psi(G)^2$. Hence $\varphi_{\{\psi I\}}(G)^2+\psi_{\{\varphi I\}}(G)^2>0$.

Suppose $G\cdot(\sum_n G_n)>0$. Let n be the smallest integer for which $G_n\cdot G>0$, and set $G_0=G_n+G$. Then $\overline{G}_0-\underline{G}_0>\overline{G}_n-\underline{G}_n$ (since G does not lie entirely in G_n) and this together with $G_0\subseteq I$, $G_0\cdot(G_1+\cdots+G_{n-1})=0$, $O_{G_0}\varphi+O_{G_0}\psi\geqq O_G\varphi_{\{\psi I\}}+O_{G_{\{\varphi I\}}}>0$ shows that $\varphi(G_0)^2+\psi(G_0)^2>0$. But $\varphi_{\{\psi I\}}(G)^2+\psi_{\{\varphi I\}}(G)^2=\varphi(G)^2+\psi(G)^2$. Hence $\varphi_{\{\psi I\}}(G)^2+\psi_{\{\varphi I\}}(G)^2>0$.

The lemma is thus completely proved.

We now define $\varphi_{\{\psi\pi\}}$, $\psi_{\{\varphi\pi\}}$ as follows:

$$\varphi_{\{\psi\pi\}}(t) = \varphi_{\{\psi\Delta\pi\}}(t) \text{ if } t \text{ is in } \Delta\pi \quad (\Delta\pi);$$

$$\psi_{\{\varphi\pi\}}(t) = \psi_{\{\varphi\Delta\pi\}}(t) \text{ if } t \text{ is in } \Delta\pi \quad (\Delta\pi).$$

It is easily seen that there is no conflict of definition at the division points of π.

We now define integration processes $\int_{\{I\}}$, $\int_{\{\pi\}}$, $\left|\int\right|$, $a\int+b\int_{\{\pi\}}$, etc., thus:

$$\int_{\{I\}}\psi\,d\varphi\equiv\int\psi_{\{\varphi I\}}\,d\varphi_{\{\psi I\}},$$

$$\int_{\{\pi\}}\psi\,d\varphi\equiv\int\psi_{\{\varphi\pi\}}\,d\varphi_{\{\psi\pi\}},$$

$$\left|\int\right|\psi\,d\varphi\equiv\left|\int\psi\,d\varphi\right|,$$

$$\left(a\int+b\int_{\{\pi\}}\right)\psi\,d\varphi\equiv a\int\psi\,d\varphi+b\int_{\{\pi\}}\psi\,d\varphi, \text{ etc.}$$

If $f(\pi)$ has a meaning for every partition π of T or of a sub-interval I, we shall denote by $\overline{\mathrm{B}}_I f$ the least upper bound of $f(\pi)$ for all partitions of I and by $\overline{\mathrm{B}}_\pi f$ the sum $\sum_{\Delta\pi}^{\pi}\overline{\mathrm{B}}_{\Delta\pi}f$.

Inequality $\{I\}$.

$$\left|\frac{1}{2}\{\psi_{\pi(I)}(\underline{I})+\psi_{\pi(I)}(\overline{I})\}\,\varphi_{\pi(I)}(I)-\int_{\{I\}I}\psi_{\pi(I)}\,d\varphi_{\pi(I)}\right|\leqq 2\,\overline{\mathrm{B}}_I\,S\,\mathrm{E}^2\,\Delta\,\alpha_{\varphi\psi}.$$

INEQUALITY $\{\pi\}$.

$$\left|\int \psi_\pi \, d\varphi_\pi - \int_{\{\pi\}} \psi_{\pi'} \, d\varphi_{\pi'}\right| \leqq 2\,\overline{\mathrm{B}}_\pi\, S\mathrm{E}^2 \Delta\, \alpha_{\varphi\psi} \quad (\pi' \mathrm{F}\, \pi).$$

It is clear that Inequality $\{\pi\}$ follows from Inequality $\{I\}$, which we now prove.

Let $t_1 < t_2 < \cdots < t_{n-1}$ be the values for which

$$\begin{vmatrix} \varphi_{\pi\{\psi_\pi I\}}(t) & \psi_{\pi\{\varphi_\pi I\}}(t) & 1 \\ \varphi_\pi(\underline{I}) & \psi_\pi(\underline{I}) & 1 \\ \varphi_\pi(\overline{I}) & \psi_\pi(\overline{I}) & 1 \end{vmatrix} = 0,$$

and π_0 the partition $\underline{I} = t_0, t_1, \cdots, t_{n-1}, t_n = \overline{I}$.

Now by the algebraic identity of § 8

$$\frac{1}{2}\{\psi_\pi(\underline{I}) + \psi_\pi(\overline{I})\}\, \varphi_\pi(I) - \int_{\{I\}I} \psi_\pi \, d\varphi_\pi$$

$$= \sum_{i=1}^{n} \left[\frac{1}{2}\{\psi_\pi(t_{i-1}) + \psi_\pi(t_i)\}\{\varphi_\pi(t_i) - \varphi_\pi(t_{i-1})\} - \int_{t_{i-1}}^{t_i} \psi_{\pi\{\varphi_\pi I\}}\, d\varphi_{\pi\{\psi_\pi I\}}\right].$$

Hence by Lemma $\mathrm{U}_0 1$

$$\left|\frac{1}{2}\{\psi_\pi(\underline{I}) + \psi_\pi(\overline{I})\}\, \varphi_\pi(I) - \int_{\{I\}I} \psi_\pi \, d\varphi_\pi\right| \leqq \sum_{\Delta\pi_0}^{\pi_0} \mathrm{E}^2\, \alpha_{\psi_\pi \varphi_\pi}(\Delta\pi_0).$$

The partition π_0 is such that between any two of its consecutive division points t_{i-1}, t_i there is a division point u_i of π. Now let π', π'' be the partitions

$$\pi': \quad \underline{I}, u_1, u_3, u_5, \cdots, \overline{I},$$

$$\pi'': \quad \underline{I}, u_2, u_4, u_6, \cdots, \overline{I}.$$

Then every cell $\Delta\pi_0$ is in a cell $\Delta\pi'$ or in a cell $\Delta\pi''$.

Therefore

$$\begin{aligned} \sum_{\Delta\pi_0}^{\pi_0} \mathrm{E}^2\, \alpha_{\varphi_\pi\psi_\pi}(\Delta\pi_0) &\leqq \sum_{\Delta\pi'}^{\pi'} \mathrm{E}^2\, \alpha_{\varphi_\pi\psi_\pi}(\Delta\pi') + \sum_{\Delta\pi''}^{\pi''} \mathrm{E}^2\, \alpha_{\varphi_\pi\psi_\pi}(\Delta\pi'') \\ &\leqq \sum_{\Delta\pi'}^{\pi'} \mathrm{E}^2\, \alpha_{\varphi\psi}(\Delta\pi') + \sum_{\Delta\pi''}^{\pi''} \mathrm{E}^2\, \alpha_{\varphi\psi}(\Delta\pi'') \\ &\leqq 2\,\overline{\mathrm{B}}_I\, S\mathrm{E}^2\, \alpha_{\varphi\psi}, \end{aligned}$$

which establishes Inequality $\{I\}$.

10. The necessary condititions J. The identity

$$\int \psi_{\pi'}\, d\varphi_{\pi'} - \int \psi_{\pi''}\, d\varphi_{\pi''} = \int \psi_{\pi'}\, d\varphi_{\pi'} - \int_{\{\pi'\}} \psi_{\pi'''}\, d\varphi_{\pi'''}$$
$$+ \int_{\{\pi'\}} \psi_{\pi'''}\, d\varphi_{\pi'''} - \int \psi_{\pi'''}\, d\varphi_{\pi'''}$$
$$+ \int \psi_{\pi'''}\, d\varphi_{\pi'''} - \int_{\{\pi''\}} \psi_{\pi'''}\, d\varphi_{\pi'''}$$
$$+ \int_{\{\pi''\}} \psi_{\pi'''}\, d\varphi_{\pi'''} - \int \psi_{\pi''}\, d\varphi_{\pi''}$$

implies the inequalities J 1, J 2.

INEQUALITY J 1.

$$\left| \int \psi_{\pi'}\, d\varphi_{\pi'} - \int \psi_{\pi''} - d\varphi_{\pi''} \right|$$
$$\leqq 2\,[\overline{\mathrm{B}}_{\pi'}\, S\mathrm{E}^2 \Delta\, \alpha_{\varphi\psi} + \overline{\mathrm{B}}_{\pi''}\, S\mathrm{E}^2 \Delta\, \alpha_{\varphi\psi}]$$
$$+ \left[\left| \int_{\{\pi'\}} \psi_{\pi'''}\, d\varphi_{\pi'''} - \int \psi_{\pi'''}\, d\varphi_{\pi'''} \right| \right.$$
$$\left. + \left| \int_{\{\pi''\}} \psi_{\pi'''}\, d\varphi_{\pi'''} - \int \psi_{\pi'''}\, d\varphi_{\pi'''} \right| \right] \qquad (\pi'''\,\mathrm{F}\,\pi',\ \pi'''\,\mathrm{F}\,\pi'').$$

INEQUALITY J 2.

$$\left| \int_{\{\pi'\}} \psi_{\pi'''}\, d\varphi_{\pi'''} - \int \psi_{\pi'''}\, d\varphi_{\pi'''} \right| + \left| \int_{\{\pi''\}} \psi_{\pi'''}\, d\varphi_{\pi'''} - \int \psi_{\pi'''}\, d\varphi_{\pi'''} \right|$$
$$\leqq \left| \int \psi_{\pi'}\, d\varphi_{\pi'} - \int \psi_{\pi''}\, d\varphi_{\pi''} \right|$$
$$+ 2\,[\overline{\mathrm{B}}_{\pi'}\, S\mathrm{E}^2 \Delta\, \alpha_{\varphi\psi} + \overline{\mathrm{B}}_{\pi''}\, S\mathrm{E}^2\, \alpha\, \Delta_{\varphi\psi}] \qquad (\pi'''\,\mathrm{F}\,\pi',\ \pi'''\,\mathrm{F}\,\pi'').$$

THEOREM N 4. *The conditions*

$$J_F(\varphi\,\psi):\ \underset{\pi'\pi''}{\mathrm{L_F}}\ \underset{\pi'''}{\underline{L}_{\mathrm{F}}} \left[\left| \int_{\{\pi'\}} - \int \right| + \left| \int_{\{\pi''\}} - \int \right| \right] \psi_{\pi'''}\, d\varphi_{\pi'''} = 0,$$

$$J_N(\varphi\,\psi):\ \underset{\pi'\pi''}{\mathrm{L_N}}\ \underset{\pi'''}{\underline{L}_{\mathrm{N}}} \left[\left| \int_{\{\pi'\}} - \int \right| + \left| \int_{\{\pi''\}} - \int \right| \right] \psi_{\pi'''}\, d\varphi_{\pi'''} = 0$$

are respectively necessary for the existence of $\int_0^1 \psi\, d\varphi$ *in the senses* (FW), (NW) *and therefore respectively necessary for its existences in the senses* (FS), (NS).

This theorem follows from inequality J 2.

11. The general existence theorem. By $S_\pi(\Delta O\psi)|\Delta\varphi|$ will be denoted $\sum_{\Delta\pi}^{\pi}(O_{\Delta\pi}\psi)|\varphi(\Delta\pi)|$, and by $\bar{\mathrm{B}}_I S(\Delta O\psi)|\Delta\varphi|$ will be denoted the least upper bound of $S_\pi(\Delta O\psi)|\Delta\varphi|$ for partitions π of I; and by $\bar{\mathrm{B}}_\pi S(\Delta O\psi)|\Delta\varphi|$ will be denoted $\sum_{\Delta\pi}^{\pi}\bar{\mathrm{B}}_{\Delta\pi}S(\Delta O\psi)|\Delta\varphi|$.

INEQUALITY S_0. $\mathrm{E}^2\alpha_{\varphi\psi}(I)\leqq 5\bar{\mathrm{B}}_I S(\Delta O\psi)|\Delta\varphi|$.

For let t_1, t_2, t_3 be taken in I so that

$$\left|\frac{1}{2}\{\psi(t_1)+\psi(t_2)\}[\varphi(t_2)-\varphi(t_1)]+\frac{1}{2}\{\psi(t_2)+\psi(t_3)\}[\varphi(t_3)-\varphi(t_2)]\right.$$
$$\left.+\frac{1}{2}\{\psi(t_3)+\psi(t_1)\}[\varphi(t_1)-\varphi(t_3)]\right|\leqq\frac{1}{5}\mathrm{E}^2\alpha_{\varphi\psi}(I).$$

Then

$$|\psi(t_1)-\psi(t_2)|\,|\varphi(t_1)-\varphi(t_2)|+|\psi(t_2)-\psi(t_3)|\,|\varphi(t_2)-\varphi(t_3)|$$
$$+|\psi(t_3)-\psi(t_1)|\,|\varphi(t_3)-\varphi(t_1)|\geqq\frac{2}{5}\mathrm{E}^2\alpha_{\varphi\psi}(I)$$

by the algebraic inequality

$$|(x_1-x_2)(y_1+y_2)+(x_2-x_3)(y_2+y_3)+(x_3-x_1)(y_3+y_1)|$$
$$\leqq|x_1-x_2|\,|y_1-y_2|+|x_2-x_3|\,|y_2-y_3|+|x_3-x_1|\,|y_3-y_1|,$$

which is easily proved.

Now let $\pi'=\underline{I}, t_1, t_2, t_3, \bar{I}$ and $\pi''=\underline{I}, t_1, t_3, \bar{I}$. Then

$$S_{\pi'}(\Delta O\psi)|\Delta\varphi|+S_{\pi''}(\Delta O\psi)|\Delta\varphi|\geqq\frac{2}{5}\mathrm{E}^2\alpha_{\varphi\psi}(I);$$

from which the inequality follows.

INEQUALITY S. $S_\pi\mathrm{E}^2\Delta\alpha_{\varphi\psi}\leqq 5\bar{\mathrm{B}}_\pi S(\Delta O\psi)|\Delta\varphi|$.

Let us introduce the conditions

O_F: $\quad\mathrm{L_F}S(\Delta O\psi)|\Delta\varphi|=0,$

O_N: $\quad\mathrm{L_N}S(\Delta O\psi)|\Delta\varphi|=0.$

We can now state the

EXISTENCE THEOREM. *The four pairs of conditions*

$A_F(\varphi\psi)$, $\quad J_F(\varphi\psi)$;
$A_N(\varphi\psi)$, $\quad J_N(\varphi\psi)$;
$J_F(\varphi\psi)$, $\quad O_F(\varphi\psi)$;
$J_N(\varphi\psi)$, $\quad O_N(\varphi\psi)$

are respectively necessary and sufficient for the existence of $\int_0^1 \psi\, d\varphi$ *in the senses* (FW), (NW), (FS), (NS).

The necessity of the various conditions has already been proved. The sufficiency of the first two pairs follows from inequality J_1 on operating on both sides of that inequality by $\underset{\pi'\pi''}{\mathrm{L_F}}\ \underset{\pi'''}{L_F}$, $\underset{\pi'\pi''}{\mathrm{L_N}}\ \underset{\pi'''}{L_N}$ respectively. The sufficiency of the last two pairs then follows from inequality S_0.

LEMMA U 1. *If* $\varphi \mathrm{U} \psi$ *on* T, *then* $J_F(\varphi\psi)$ *and* $J_N(\varphi\psi)$.

LEMMA U 2. *If* $\alpha_{\varphi\psi}(T)$ *is a continuous arc with at most a finite number of multiple points, then* $\varphi \mathrm{U} \psi$ *on* T.

This is easily shown by slightly modifying a proof of de la Vallée Poussin (see Pierpont, *Theory of Functions of Real Variables*, vol. II, p. 597).

LEMMA U 3. *If* φ *is monotone on* T, *then* $\varphi \mathrm{U} \psi$.

COROLLARY 1. *The four conditions* A_F, A_N, O_F, O_N *are respectively necessary for the existence of* $\int_0^1 \psi\, d\varphi$ *in the senses* (FW), (NW), (FS), (NS); *and are respectively sufficient if* $\varphi \mathrm{U} \psi$ *on* T, *in particular, if* $\alpha_{\varphi\psi}(T)$ *is a continuous curve with a finite number of multiple points, or if* φ *is a monotone function on* T.

Suppose φ is of limited variation on T, that is, that $\int_0^1 |d\varphi|$ exists. Then $\int_I |d\varphi|$ exists for every I in T and will be denoted by $\mathrm{V}_\varphi(I)$. We define two functions $\varphi_1(t)$, $\varphi_2(t)$ on T by the equations

$$\varphi_1(t) \equiv \frac{1}{2}\left\{\int_0^t |d\varphi| + [\varphi(t)-\varphi(0)]\right\} + \varphi(0),$$

$$\varphi_2(t) \equiv \frac{1}{2}\left\{\int_0^t |d\varphi| - [\varphi(t)-\varphi(0)]\right\},$$

so that

$$\varphi(t) = \varphi_1(t) - \varphi_2(t)$$

and

$$\Delta\varphi_1 = \frac{1}{2}\{\Delta \mathrm{V}_\varphi + \Delta\varphi\},$$

$$\Delta\varphi_2 = \frac{1}{2}\{\Delta \mathrm{V}_\varphi - \Delta\varphi\}.$$

Hence φ_1, φ_2 are monotonic increasing functions in view of the obvious inequality

$$|\Delta\varphi| \leqq \Delta \mathrm{V}_\varphi.$$

We have, if $O_T\psi$ is finite,

$$\mathrm{L_F}\, S(\Delta O\psi)\{|\Delta\varphi| - \Delta \mathrm{V}_\varphi\} = \mathrm{L_N}\, S(\Delta O\psi)\{|\Delta\varphi| - \Delta \mathrm{V}_\varphi\} = 0;$$

from which it follows that the conditions

$$V_F(\varphi\psi): \qquad \mathrm{L}_F\, S(\Delta O\psi)\Delta \mathrm{V}_\varphi = 0,$$
$$V_N(\varphi\psi): \qquad \mathrm{L}_N\, S(\Delta O\psi)\Delta \mathrm{V}_\varphi = 0$$

are respectively equivalent to the conditions $O_F(\varphi\psi)$, $O_N(\varphi\psi)$, so that $V_F(\varphi\psi)$, $V_N(\varphi\psi)$ are necessary conditions for the existence of $\int_0^1 \psi d\varphi$ in the senses (FS), (NS) respectively. Moreover the equations

$$\Delta \mathrm{V}_\varphi = \Delta\varphi_1 + \Delta\varphi_2,$$
$$\Delta\varphi = \Delta\varphi_1 - \Delta\varphi_2$$

show that the conditions $V_F(\varphi\psi)$, $V_N(\varphi\psi)$ imply respectively the pair of conditions $V_F(\varphi_1\psi)$, $V_F(\varphi_2\psi)$ and $V_N(\varphi_1\psi)$, $V_N(\varphi_2\psi)$. Hence the conditions $V_F(\varphi\psi)$, $V_N(\varphi\psi)$ imply the existence of $\int_0^1(\psi d\varphi)$ in the respective senses (FS), (NS) with the value

$$\int_0^1 \psi d\varphi = \int_0^1 \psi d\varphi_1 - \int_0^1 \psi d\varphi_2.$$

We summarize these well known results as

COROLLARY 2. *If φ is of limited variation and $O_T\psi$ is finite, either of the conditions $V_F(\varphi\psi)$, $O_F(\varphi\psi)$ is necessary and sufficient for the existence of $\int_0^1 \psi d\varphi$ in the sense (FS) and either of the conditions $V_N(\varphi\psi)$, $O_N(\varphi\psi)$ is necessary and sufficient for the existence of $\int_0^1 \psi d\varphi$ in the sense (NS).*

12. **A squarable crinkly curve whose associated Stieltjes integral fails to exist.** If $P_1, \cdots, P_n$ are any n points in a plane in which a system of rectangular coördinates has been established, let $(P_1, \cdots, P_n)$ be defined by the equation

$$(P_1, \cdots, P_n) = \sum_{i=1}^{n} \frac{1}{2}\{\text{ord } P_i + \text{ord } P_{i+1}\}\ [\text{abs } P_{i+1} - \text{abs } P_i].$$

Now let S denote a square of which two sides are parallel to and above the x-axis. Let us agree to denote the area of any geometric figure by the same letter as the figure so that S will also denote the area of the square S. Let AB represent one diagonal of S. Finally let f be any positive integer.

Take a positive integer p. Divide S into p^2 equal squares. Then divide each of these squares into p^2 equal squares, and so on. In this way we secure an infinite sequence of divisions of S into p^2, p^4, $p^6, \cdots$ equal squares. The vertices of these squares form a set $[X]$ everywhere dense in S. The number m such that p^{2m} is the smallest number (a power of p^2) of equal squares into which S may be divided so that X appears as a vertex is called the order of X.

Denote by M that one of the vertices of S other than A and S for which it is true that $(ABMA)$ is positive.

Let $A_1 B_1, \cdots, A_r B_r$ be r sub-segments of the segment AB with end points in $[X]$ and such that A_{i+1} is between $A_i (i = 1, \cdots, r)$ and any B_j $(j = 1, \cdots, r)$, and B_{i+1} is between B_i and any A_j and such moreover that

$$\overline{A_1 B_1}^2 + \cdots + \overline{A_r B_r}^2 > 2f \cdot \overline{AB}^2.$$

Now let $M_1, \cdots, M_r$ be r points all on the same side of AB as M and such that $A_i M_i B_i$ is a right angle $(i = 1, \cdots, r)$. Then the points M_i are in $[X]$, the quantities $(A_i B_i M_i A_i)$ are positive, and the broken lines $A_i M_i B_i$ do not have any points in common.

Next take any point A_0 of the set $[X]$ which is within the segment AA_1 and then choose $N_1, \cdots, N_r$ all on the opposite side of AB from M so that the angles $A_{i-1} N_i B_i$ shall all be right angles. Then $N_1, \cdots, N_r$ are in $[X]$, the quantities $(A_{i-1} N_i B_i A_{i-1})$ are positive, and the broken lines $A_{i-1} N_i B_i$, $B_j M_j A_j$ have no points (other than end points) in common.

Thus the points

$$AA_0,\ N_1 B_1 M_1 A_1, \cdots,\ N_i B_i M_i A_i, \cdots,\ N_r B_r M_r A_r$$

taken in order form the vertices in $[X]$ of a simple broken line which joins A to A_r, and consists of segments each of which, except AA_0, is parallel to a coördinate axis. It is clearly possible to join A_r to B by a broken line of the same character which does not have any point other than A_r in common with this broken line. Let $A_r Q_1 \cdots Q_s B$ denote such a broken line. Then

$$AA_0 N_1 B_1 M_1 A_1 \cdots N_i B_i M_i A_i \cdots N_r B_r M_r A_r Q_1 \cdots Q_s B$$

taken in order are the vertices of a simple broken line $A\lambda_0 B$ which consists, aside from the segments AA_0 and $Q_s B$, entirely of segments parallel to the axes of coördinates.

The points

$$AB_1 M_1 A_1 \cdots B_i M_i A_i \cdots B_r M_r A_r B$$

taken in order form the vertices of a broken line $A\lambda' B$ which is inscribed in $A\lambda_0 B$.

If we note that for any three collinear points PQR it is true that $(PQR) = (PQ) + (QR)$, then we see that

$$\begin{aligned}(A\lambda' B) &= (AA_1) + (A_1 B_1 M_1 A_1) + (A_1 A_2) + (A_2 B_2 M_2 A_2) + \cdots \\ &\quad + (A_{i-1} A_i) + (A_i B_i M_i A_i) + \cdots \\ &\quad + (A_{r-1} A_r) + (A_r B_r M_r A_r) + (A_r B) \\ &= (AB) + \sum_{i=1}^{r} (A_i B_i M_i A_i) > (AB) + fS,\end{aligned}$$

so that

$$(A\lambda' B) - (AB) > fS.$$

The vertices of $A\lambda_0 B$ are all in $[X]$ and hence there is a finite least upper bound k_0 for their orders. Let l be the length of $A\lambda_0 B$. Take $k = k_0 + 2$ so that

$$l \frac{\sqrt{S}}{p^k} < \frac{S}{2} \quad \text{and} \quad < (A\lambda' B) - (AB) - fS.$$

Now suppose S divided into p^{2k} equal squares. Shade all of these squares which have a side in common with a segment of $A\lambda_0 B$ and which lie on the same side of $A\lambda_0 B$ as M does. Let us now suppose that p is even. Then there is an even number of shaded squares against each segment of $A\lambda_0 B$ except AA_0 and $Q_s B$. With this exception, then, it is possible to replace each segment of $A\lambda_0 B$ by a broken line joining the end points of that segment and made up by taking one diagonal from each of the shaded squares that abut thereon. After this has been done and all segments deleted which enter twice in opposite senses, there is obtained a simple broken line $A\lambda B$ all of whose segments, except AA_0 and $Q_s B$, are diagonals of shaded squares. This exception can be removed by shading also all the squares of our division which have an interior point in common with AA_0 or $Q_s B$ and then regarding all vertices of these new shaded squares which lie on AA_0 and on BQ_s as vertices of $A\lambda B$. It is clear that $A\lambda' B$ is inscribed in $A\lambda B$.

We have thus shown how to replace the diagonal AB of S by a simple broken line $A\lambda B$ subject to the following conditions:

(A) the broken line $A\lambda B$ consists of diagonals of certain shaded squares of a division of S into equal squares;

(B) the squares of which the segments of $A\lambda B$ are diagonals have total area less than $S/2$;

(C) a broken line $A\lambda' B$ can be inscribed in $A\lambda B$ for which

$$(A\lambda' B) - (AB) > fS.$$

We can now construct our crinkly curve.

For the sake of being definite, let S be the unit square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ and A, B be the points (0,0), (1,1), respectively. Then on setting $\lambda_0 = AB$, $S_0 = S$, there is for every sequence $f_1, f_2, \cdots$ of positive numbers a sequence $A\lambda_0 B$, $A\lambda_1 B$, $A\lambda_2 B, \cdots$ of simple broken lines joining A to B and subject to the following conditions:

(A) every segment of $A\lambda_n B$ is a diagonal of a square of a certain division of S into equal squares;

(B) the total area S_{n+1} of the squares whose diagonals are the segments of $A\lambda_{n+1} B$ is less than half the total area S_n of the squares S_n whose diagonals are the segments of $A\lambda_n B$;

(C) every vertex of $A\lambda_n B$ is a vertex of $A\lambda_{n+1} B$;

(D) in $A\lambda_{n+1} B$ may be inscribed a broken line $A\lambda'_{n+1} B$ such that

$$(A\lambda'_{n+1} B) - (A\lambda_n B) > f_{n+1} \cdot S_n.$$

To obtain such a sequence we have only to apply the process above, first to the diagonal of the square S, second to each segment of the broken line $A\lambda_1 B$ so obtained, and so on, at each step taking the proper value for f.

Let

$$A\lambda_n B: \qquad x = \varphi_n(t), \qquad y = \psi_n(t) \qquad\qquad (0 \leqq t \leqq 1)$$

denote a one-to-one representation of $A\lambda_n B$ on to $T = (0, 1)$ such that two equal sub-segments of $A\lambda_n B$ always correspond to two equal sub-intervals of T. Then φ_n, ψ_n converge to two continuous functions φ, ψ, respectively, since φ_n, ψ_n are continuous and the convergence is uniform.

The arc

$$\Gamma: \qquad x = \varphi(t), \qquad y = \psi(t) \qquad\qquad (0 \leqq t \leqq 1)$$

is a simple continuous arc joining A to B.

The broken lines $A\lambda_n B$ (and hence the broken lines $A\lambda'_n B$) are inscribed in Γ. Moreover a vertex of $A\lambda_n B$ is given by the same value of the parameter t in the equations of $A\lambda_n B$ as in the equations of Γ.

The arc Γ is *squarable*. For it lies entirely in the squares S_n and these have, by (B) above, total area less than $1/2^n$.

Now suppose the numbers $f_1 S_0$, $f_2 S_1$, $f_3 S_2$, $\cdots$ are all bounded from zero, say all greater than e_0. This is possible since the choice of the f's is absolutely arbitrary. Then

$$(1) \qquad (A\,\lambda'_{n+1}\,B) - (A\,\lambda_n\,B) > e_0 \qquad (n = 0, 1, 2, \cdots).$$

The Stieltjes integral $\int_0^1 \psi(t)\,d\varphi(t)$ *does not exist in either of the four senses.* For first the vertices of $A\,\lambda_n\,B$ and $A\,\lambda'_{n+1}\,B$ correspond to two divisions of T of norm in each case certainly less than $1/p^n$. Moreover $(A\,\lambda'_{n+1}\,B)$, $(A\,\lambda_n\,B)$ represent two sums of the form

$$S^0\,\psi\,\Delta\,\varphi$$

corresponding to those divisions. These facts together with the inequality (1) above show the required non-existence.

13. **On the independence of the sufficient conditions.** In this section we find functions φ, ψ for which $A_F(\varphi, \psi)$, $A_N(\varphi, \psi)$, $O_F(\varphi, \psi)$, $O_N(\varphi, \psi)$ but not $J_N(\varphi, \psi)$ or $J_F(\varphi, \psi)$.

Let a square S be divided into $(2\,p+1)^2$ equal squares. Let AB be opposite vertices and represent the remaining vertices by MN in such a way that $(AMBA) > 0$. The diagonal AB is divided by the network of $(2\,p+1)^2$ equal squares into $2\,p+1$ equal segments whose end points we will denote in order by

$$A\,A_1\,A_2 \cdots A_p\,B_p\,B_{p-1} \cdots B_1\,B.$$

Now take $M_1 \cdots M_p N_1 \cdots N_p$ so that $A_i M_i B_i N_i$ is a square S_i $(i = 1, \cdots, p)$ and M_i, N_i are on the same sides of AB as M, N respectively. Set $A_0 = A$, $B_0 = B$, $S_0 = S$. Denote the squares whose diagonals are respectively $A_i\,A_{i+1}$, $B_i\,B_{i+1}$, $A_p\,B_p$ $(i = 0, \cdots, p-1)$ by σ'_i, σ''_i, σ'_p or σ''_p.

There is a simple broken line $A_i\,\lambda'_i\,B_i$ which consists of diagonals of squares (of our network) which lie in $S_{i-1} - (S_i + \sigma'_{i-1} + \sigma''_{i-1})$ on the same side of AB as M and a simple broken line $B_i\,\lambda''_i\,A_i$ consisting of diagonals of squares which lie in $S_{i-1} - (S_i + \sigma'_{i-1} + \sigma''_{i-1})$ on the same side of AB as N.

Now let $A\lambda B$ be the broken line

$$A\,A_1\,\lambda'_1 B_1\,\lambda''_1 A_1 A_2\,\lambda'_2 B_2 \lambda''_2\,A_2 \cdots A_p\,\lambda'_p B_p\,\lambda''_p\,A_p\,B_p\,B_{p-1} \cdots B_1\,B.$$

Then the segments of $A\lambda B$ are diagonals of the $(2p+1)^2$ equal squares into which S has been divided. Moreover

$$
\begin{aligned}
(A\lambda B &= (AA_1) + (A_1\lambda_1' B_1 \lambda_1'' A_1) + (A_1 A_2) + (A_2 \lambda_2' B_2 \lambda_2'' A_2) + \cdots \\
&\qquad + A_p \lambda_p' B_p \lambda_p'' A_p) + (A_p B_p B_{p-1} \cdots B_1 B) \\
&= (AB) + \sum_{i=1}^{p} (A_i \lambda_i' B_i \lambda_i'' A_i).
\end{aligned}
$$

But

$$
\begin{aligned}
(A_p \lambda_p' B_p \lambda_p'' A_p) &= \frac{2^2}{(2p+1)^2} S, \\
(A_{p-1} \lambda_{p-1}' B_{p-1} \lambda_{p-1}'' A_{p-1}) &= \frac{4^2}{(2p+1)^2} S, \\
\cdots\cdots & \cdots\cdots \\
(A_1 \lambda_1' B_1 \lambda_1'' A_1) &= \frac{(2p)^2}{(2p+1)^2} S.
\end{aligned}
$$

Hence

$$
(A\lambda B) - (AB) = \frac{1}{(2p+1)^2} \{2^2 + 4^2 + \cdots + (2p)^2\} S,
$$

or

$$
(A\lambda B) - (AB) = \frac{2p\,(p+1)}{3\,(2p+1)} S.
$$

Let S be the unit square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ and A, B the points $(0\,0)$, $(1\,1)$ respectively. Let $\{p_n\}$ be any infinite sequence of integers. Set $q_n = 2\,(2p_n + 1)\,(p_n + 1)$, $m_n = q_1 \cdots q_n$, $r_n = 2p_n + 1$ for $n = 1, 2, \cdots$.

Set $P_{00} = A$, $P_{0m_0} = P_{01} = B$, and let M_{01} be the mid-point of $P_{00} P_{01}$.

There is a broken line $P_{10} \cdots P_{1m_1}$ such that

(1_1) $P_{10} = A$, $P_{1m_1} = B$;

(2_1) the points $P_{10}, \cdots, P_{1r_1 - 1}$ divide $P_{10} P_{1r_1}$ or $P_{00} M_{01}$ into r_1 equal parts;

(3_1) the points $P_{1r_1}, \cdots, P_{1q_1}$ $(= P_{1m_1})$ are the vertices of a broken line which consists of diagonals of the r_1^2 equal squares into which square $M_{01} P_{01}$ may be divided and which is such that

$$
(P_{1r_1} \cdots P_{1q_1}) - (M_{01} P_{01}) = \frac{2p_1\,(p_1 + 1)}{3\,(2p_1 + 1)} \cdot \text{sq.} M_{01} P_{01}.
$$

On continuing in this way we may construct a sequence of broken lines $P_{n0} \cdots P_{nm_n}$ $(n = 1, 2, \cdots)$ such that

(1_n)
$$
P_{n,kq_n} = P_{n-1,k} \qquad (k = 0, \cdots, m_{n-1});
$$
$$
P_{n,kq_n + r_n} = M_{n-1,k+1},
$$

where $M_{n-1,k+1}$ is the mid-point of $P_{n-1,k} P_{n-1,\,k+1}$;

(2_n) the points $P_{n,kq_n+1},\cdots, P_{n,kq_n+r_n-1}$ divide $P_{n,kq_n} P_{n,kq_n+r_n}$ into r_n equal parts;

(3_n) the points $P_{n,kq_n+r_n},\cdots, P_{n,(k+1)q_n}$ are the vertices of a broken line whose segments are diagonals of the r_n^2 equal squares into which square $P_{n,kq_n+r_n} P_{n,(k+1)q_n}$ $(= M_{n-1,k+1} P_{n-1,k+1})$ may be divided and which is such that

$$(P_{n,kq_n+r_n},\cdots, P_{n,(k+1)q_n})-(M_{n-1,k+1}, P_{n-1,k+1})$$
$$=\frac{2p_n(p_n+1)}{3(2p_n+1)}\cdot \text{sq.}\, M_{n-1,k+1} P_{n-1,k+1}.$$

Let Σ_n denote the area of the squares whose diagonals are the segments of $(P_{n0}\cdots P_{nm_n})$. Then

$$\Sigma_n=\frac{m_n}{(4p_1+2)^2\cdots(4p_n+2)^2}=\frac{(p_1+1)\cdots(p_n+1)}{(4p_1+2)\cdots(4p_n+2)},$$

or

$$\Sigma_n=\frac{1}{4^n}\left(1+\frac{1}{2p_1+1}\right)\cdots\left(1+\frac{1}{2p_n+1}\right).$$

Hence

$$\left(\frac{1}{4}\right)^n<\ \Sigma_n<\left(\frac{1}{2}\right)^n.$$

We have

$$(P_{n0},\cdots, P_{nm_n})-(P_{n-1,0},\cdots, P_{n-1,m_{n-1}})$$
$$=\sum_{k=0}^{m_{n-1}-1}\{(P_{n,kq_n},\cdots, P_{n,(k+1)q_n})-(P_{n-1,k} P_{n-1,k+1})\},$$

so that

$$(P_{n0}\cdots P_{nm_n})-(P_{n-1,0}\cdots P_{n-1,m_{n-1}})=\frac{p_n(p_n+1)}{6(2p_n+1)}\ \Sigma_{n-1}.$$

Let l_n be the length of $P_{n0}\cdots P_{nm_n}$. Then

$$l_n=\frac{\sqrt{2}\, m_n}{(4p_1+2)\cdots(4p_n+2)}=\sqrt{2}\,(p_1+1)\cdots(p_n+1).$$

The sequence $\{p_n\}$ has so far been arbitrary. Now take p_n to be 1 plus the greatest integer in $12(4^{n-1}/n)$. Then

$$\frac{4^{n-1}}{n}\leqq\frac{p_n(p_n+1)}{6(2p_n+1)}\leqq\frac{1}{8}+\frac{4^{n-1}}{n}.$$

Hence

$$(P_{n0}\cdots P_{nm_n})-(P_{n-1,0}\cdots P_{n-1,m_{n-1}})\geqq\frac{4^{n-1}}{n}\cdot\left(\frac{1}{4}\right)^{n-1}=\frac{1}{n},$$

and

$$(P_{n0}\cdots P_{nm_n})-(P_{n-1,0}\cdots P_{n-1,m_{n-1}})$$
$$<\left(\frac{4^{n-1}}{n}+\frac{1}{8}\right)\left(\frac{1}{4}\right)^{n-1}\left(1+\frac{1}{2p_1+1}\right)\cdots\left(1+\frac{1}{2p_{n-1}+1}\right)$$
$$<\left(\frac{1}{n}+\frac{1}{2}\cdot\frac{1}{4^n}\right)\left(1+\frac{1}{2p_1+1}\right)\cdots\left(1+\frac{1}{2p_{n-1}+1}\right).$$

The infinite product

$$\left(1+\frac{1}{2p_1+1}\right)\left(1+\frac{1}{2p_2+1}\right)\cdots$$

converges, since the series

$$\frac{1}{2p_1+1}+\frac{1}{2p_2+1}+\cdots$$

converges, being dominated by the convergent series

$$\frac{1}{24}\left(1+\frac{2}{4}+\frac{3}{4^2}+\frac{4}{4^3}+\cdots\right).$$

Hence

$$\underset{n\to\infty}{\mathrm{L}}\{(P_{n0}\cdots P_{nm_n})-(P_{n-1,0}\cdots P_{n-1,m_{n-1}})\} = 0.$$

We note further that for every n and p,

$$(2)\qquad (P_{n+p,0}\cdots P_{n+p,m_{n+p}})-(P_{n0}\cdots P_{nm_n})\geqq\frac{1}{n+1}+\cdots+\frac{1}{n+p}.$$

Now let us write the parametric equations for the broken line $(P_{n0}\cdots P_{nm_n})$ thus:

$$x = \varphi_n(t), \qquad y = \psi_n(t) \qquad\qquad (0\leqq t\leqq 1)$$

where the parameter t is proportional to the length of arc along the broken line.

The functions $\varphi_n(t)$, $\psi_n(t)$ will converge uniformly to two continuous functions $\varphi(t)$, $\psi(t)$ respectively; it is the Stieltjes integral $\int_0^1\psi(t)\,d\varphi(t)$ of these functions which we wish to study.

Let π_n be that partition obtained by dividing (0 1) into m_n equal parts, so that $\mathrm{N}\pi_n = 1/m^n$. Then

$$\int_0^1\psi_{\pi_n}\,d\varphi_{\pi_n} = \int_0^1\psi_n\,d\varphi_n = (P_{n0}\cdots P_{nm_n}).$$

Hence by (2) $\underset{n}{\mathrm{L}}\int_0^1 \psi_{\pi_n}\, d\varphi_{\pi_n}$ does not exist and $\int_0^1 \psi\, d\varphi$ does not exist in either of the four senses.

We now investigate the conditions $O_F(\varphi\,\psi)$, $O_N(\varphi\,\psi)$. To this end, let $n(I)$ be defined by the inequality

$$\mathrm{N}\,\pi_{n(I)} \leqq I < \mathrm{N}\,\pi_{n(I)-1}.$$

We have

$$(O_I\,\psi)\,|\varphi(I)| \leqq \frac{1}{2} I_*^2\, l_{n(I)}^2,$$

where I_* is the sum of all the cells of $\pi_{n(I)}$ which have inner points in common with I. For the points $(\varphi(t), \psi(t))$ for t in I lie entirely in certain squares of the system $\Sigma_{n(I)}$. The left hand side of the inequality will have its greatest value when these squares are diagonally collinear, in which case they all lie in a square whose diagonal is $I_*\, l_{n(I)}$.

Now $I_* \leqq 3I$ always, hence

$$(O_I\,\psi)\,|\varphi(I)| \leqq \frac{9}{2} I^2\, l_{n(I)}^2.$$

Moreover

$$l_n^2 = 2\,\Sigma_n\, m_n = \frac{2\,\Sigma_n}{N\pi_n},$$

in view of which

$$(O_I\,\psi)|\varphi(I)| \leqq 9\, I^2 \frac{\Sigma_n}{N\pi_n},$$

so that

$$(O_I\,\psi)\,|\varphi(I)| \leqq 9\,[2p_{n(I)}+1]\, I\,\Sigma_{n(I)} \qquad (I \leqq [2p_{n(I)}+1]\mathrm{N}\,\pi_{n(I)}).$$

Now consider the set of all I for which $n(I) = n_0$. Since $I < \mathrm{N}\,\pi_{n_0-1}$, the points $(\varphi(t), \psi(t))$ for t in I lie entirely in two squares of Σ_{n_0-1} and hence $(O_I\,\psi)\,|\varphi(I)|$ is less than the area of four squares of Σ_{n_0-1}, that is, than $(4/(m_{n_0-1}))\,\Sigma_{n_0-1}$. By property (2_n) of the polygon $(P_{n_0} \cdots P_{nm_n})$, there is an $I_0 = (2p_{n_0}+1)\,\mathrm{N}\,\pi_{n_0}$ such that

$$(O_{I_0}\,\psi)\,|\varphi(I_0)| = \frac{1}{m_{n_0-1}}\,\Sigma_{n_0-1}.$$

Hence

$$\begin{aligned}(O_I\,\psi)\,|\varphi(I)| &\leqq 4\,(O_{I_0}\,\psi)\,|\varphi(I_0)| \\ &\leqq 36\,(2\,p_{n_0}+1)\, I_0\, \Sigma_{n_0} \\ &< 36\,(2\,p_{n_0}+1)\, I\, \Sigma_{n_0} \quad \text{if } I > (2\,p_{n_0}+1)\,\mathrm{N}\,\pi_{n_0}.\end{aligned}$$

From this it follows that for all I,

$$(O_I\psi)\,|\varphi(I)| \leqq 36\,(2\,p_{n(I)}+1)\,I\,\Sigma_{n(I)}$$
$$= \frac{216\,(p_{n(I)+1}+1)}{p_{n(I)+1}\,(p_{n(I)+1}+1)}\,I\left[\int_0^1 \psi_{n(I)+1}\,d\varphi_{n(I)+1} - \int_0^1 \psi_{n(I)}\,d\varphi_{n(I)}\right]$$
$$\leqq IM\left[\int_0^1 \psi_{n(I)+1}\,d\varphi_{n(I)+1} - \int_0^1 \psi_{n(I)}\,d\varphi_{n(I)}\right],$$

where M is the least upper bound of

$$\frac{216\,(2\,p_n+1)}{p_n\,(p_n+1)},$$

which is surely finite since

$$\underset{n}{\mathrm{L}}\,\frac{216\,(2\,p_n+1)}{p_n\,(p_n+1)} = 0.$$

Now denote by a_π the largest value of

$$\int_0^1 \psi_{n(\Delta\pi)+1}\,d\varphi_{n(\Delta\pi)+1} - \int_0^1 \psi_{n(\Delta\pi)}\,d\varphi_{n(\Delta\pi)}$$

for all $\Delta\pi$ of π.

Clearly

$$\underset{\pi}{\mathrm{L_F}}\,a_\pi = \underset{\pi}{\mathrm{L_N}}\,a_\pi = 0.$$

Hence since

$$\sum_{\Delta\pi}^{\pi} (O_{\Delta\pi}\,\psi)\,|\varphi(\Delta\pi)| \leqq M a_\pi,$$

we have

$$\mathrm{L_N}\,S(\Delta\,O\,\psi)\,|\Delta\varphi| = \mathrm{L_F}\,S(\Delta\,O\,\psi)\,|\Delta\varphi| = 0.$$

That is, φ, ψ satisfy the condition $A_F(\varphi\psi)$, $A_N(\varphi\psi)$, $O_F(\varphi\psi)$, $O_N(\varphi\psi)$, and since the integral fails to exist, do not satisfy the conditions $J_F(\varphi\psi)$, $J_N(\varphi\psi)$.

University of Minnesota,
Minneapolis, Minn.